Appendix A

Magnetic boundary conditions

Here we derive the boundary conditions on $\boldsymbol{B}$ for perfectly electrically conducting and perfectly electrically insulating boundary surfaces.

A.1 Perfectly conducting boundaries (tied field lines)

Our preferred boundary condition at the bottom of the layer is to imagine that the material below the layer is a perfect electrical conductor. We can then use the condition that $\boldsymbol{E} = 0$ within a perfect conductor. The tangential components of $\boldsymbol{E}$ are always continuous across a boundary between two materials, so E_x and E_y must both be zero at the bottom of the fluid layer. Now, Ohm's law tells us that

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{u} \wedge \boldsymbol{B}) \tag{A.1}$$

while one of Maxwell's equations gives us (neglecting the displacement current)

$$\boldsymbol{j} = \frac{1}{\mu_0} \nabla \wedge \boldsymbol{B}. \tag{A.2}$$

Equating these two expressions for $\boldsymbol{j}$, and setting E_x and E_y to zero, gives

$$u_y B_z = \eta \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) \tag{A.3}$$

$$-u_x B_z = \eta \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) \tag{A.4}$$

(where we have also used that $u_z = 0$ on the boundary). Here $\eta = 1/(\mu_0 \sigma)$ as usual. Note that after non-dimensionalizing, η becomes ζ (in the scaling used in Chapter 2) or $\zeta_0 \kappa$ (in the scaling used in Chapter 5).

A.2 Perfectly insulating boundaries (matching to a potential field)

Above the layer, we wish to match our magnetic field onto a potential field, which is equivalent to assuming that the material above the layer is perfectly insulating, with $\boldsymbol{j} = 0$. This implies that $\nabla \wedge \boldsymbol{B} = 0$, or equivalently $\boldsymbol{B} = \nabla \Omega$ for some scalar Ω (the 'potential').

Assume that the top of our layer is at $z = 0$, with z increasing downwards. Suppose that $\boldsymbol{B} = \boldsymbol{B}^+$ (a known function) within $z > 0$, and $\boldsymbol{B} = \boldsymbol{B}^-$ (to be determined) in $z < 0$. We assume periodic boundary conditions in the horizontal, so that $\boldsymbol{B}^\pm$ can be expanded as Fourier series, as follows:

$$\boldsymbol{B}^\pm = \sum_{m,n} \tilde{\boldsymbol{B}}^\pm_{mn}(z) \mathrm{e}^{\mathrm{i}\boldsymbol{k}_{mn}\cdot\boldsymbol{x}} \tag{A.5}$$

(real part understood). Here m and n are non-negative integers which number the different Fourier modes, with corresponding wavevectors $\boldsymbol{k}_{mn} = (k_{mn}, l_{mn}, 0)$.

We now consider the form of $\boldsymbol{B}^-$. This must satisfy the following:

$$\nabla \cdot \boldsymbol{B}^- = 0 \tag{A.6}$$

$$\nabla \wedge \boldsymbol{B}^- = 0 \tag{A.7}$$

$$\boldsymbol{B}^- \to \boldsymbol{B}_\infty \quad \text{as} \quad z \to -\infty \tag{A.8}$$

where $\boldsymbol{B}_\infty$ is a constant. The first two of these are just Maxwell's equations (with $\boldsymbol{j} = 0$), and the third is an additional boundary condition at infinity, which we will require for uniqueness.

We now show that (A.6)–(A.8) are satisfied if and only if $\boldsymbol{B}^-$ takes the following form:

$$\boldsymbol{B}^- = \sum_{(m,n)\neq(0,0)} \boldsymbol{\beta}_{mn} \mathrm{e}^{\mathrm{i}\boldsymbol{k}_{mn}\cdot\boldsymbol{x}} \mathrm{e}^{|\boldsymbol{k}_{mn}|z} + \boldsymbol{B}_\infty, \tag{A.9}$$

where the $\boldsymbol{\beta}_{mn}$ are defined by

$$\boldsymbol{\beta}_{mn} = C_{mn} \begin{pmatrix} \mathrm{i}k_{mn}/|\boldsymbol{k}_{mn}| \\ \mathrm{i}l_{mn}/|\boldsymbol{k}_{mn}| \\ 1 \end{pmatrix}, \tag{A.10}$$

for (arbitrary) complex constants C_{mn}.

It can be shown trivially that (A.9)–(A.10) imply (A.6)–(A.8), simply by substituting the one into the other.

To show (A.6)–(A.8) imply (A.9)–(A.10), we must solve the former for $\boldsymbol{B}^-$. Equation (A.5) gives:

$$\boldsymbol{B}^- = \sum_{m,n} \tilde{\boldsymbol{B}}^-_{mn}(z)\mathrm{e}^{\mathrm{i}\boldsymbol{k}_{mn}\cdot\boldsymbol{x}}. \tag{A.11}$$

Now

$$\nabla\cdot\boldsymbol{B}^- = 0,\ \nabla\wedge\boldsymbol{B}^- = 0 \ \Rightarrow\ \nabla^2\boldsymbol{B}^- = 0 \tag{A.12}$$

$$\Rightarrow\ -|\boldsymbol{k}_{mn}|^2 + \frac{\mathrm{d}^2\tilde{\boldsymbol{B}}^-_{mn}}{\mathrm{d}z^2} = 0 \tag{A.13}$$

$$\Rightarrow\ \tilde{\boldsymbol{B}}^-_{mn} = \boldsymbol{\alpha}_{mn}\mathrm{e}^{-|\boldsymbol{k}_{mn}|z} + \boldsymbol{\beta}_{mn}\mathrm{e}^{|\boldsymbol{k}_{mn}|z} \tag{A.14}$$

where $\boldsymbol{\alpha}_{mn}$ and $\boldsymbol{\beta}_{mn}$ are constants of integration. These equations apply for all $(m,n)\neq(0,0)$. Applying the boundary condition that $\boldsymbol{B}^-$ is bounded as $z\to-\infty$, we conclude that $\boldsymbol{\alpha}_{mn}=0$. Meanwhile for $(m,n)=(0,0)$ (i.e. the constant term in the Fourier series) we find $\tilde{\boldsymbol{B}}^-_{00} = \boldsymbol{\gamma}z+\boldsymbol{\delta}$, and from the boundary conditions, $\boldsymbol{\gamma}=0$ and $\boldsymbol{\delta}=\boldsymbol{B}_\infty$. Hence $\boldsymbol{B}^-$ is of the form (A.9), but we have yet to show that $\boldsymbol{\beta}_{mn}$ has the required form. This can be done by applying $\nabla\wedge\boldsymbol{B}^-=0$ to obtain the following equations:

$$\mathrm{i}l_{mn}\beta_{z,mn} = |\boldsymbol{k}_{mn}|\beta_{y,mn} \tag{A.15}$$

$$\mathrm{i}k_{mn}\beta_{z,mn} = |\boldsymbol{k}_{mn}|\beta_{x,mn} \tag{A.16}$$

$$\mathrm{i}k_{mn}\beta_{y,mn} = \mathrm{i}l_{mn}\beta_{x,mn}. \tag{A.17}$$

Upon solving these we find that the $\boldsymbol{\beta}_{mn}$ are as defined in (A.10), as required.

Now that we know the form of $\boldsymbol{B}^-$, we can apply boundary conditions to match $\boldsymbol{B}^-$ and $\boldsymbol{B}^+$ together. We know that B_z will be continuous across the boundary; B_x and B_y will also be continuous if there is no surface current (which is expected to be the case at an insulating boundary). Furthermore, since $\nabla\cdot\boldsymbol{B}=0$, we infer that $\mathrm{d}B_z/\mathrm{d}z$ will also be continuous across the boundary.

For $(m,n)\neq(0,0)$, continuity of B_z tells us that

$$C_{mn} = \tilde{B}^-_{z,mn} = \tilde{B}^+_{z,mn}. \tag{A.18}$$

Then from the continuity conditions $B^+_x = B^-_x$, $B^+_y = B^-_y$ and $\mathrm{d}B^+_z/\mathrm{d}z = \mathrm{d}B^-_z/\mathrm{d}z$ (all evaluated at $z = 0$), we derive the boundary conditions:

$$\tilde{B}_{x,mn} = \frac{\mathrm{i}k_{mn}}{|\boldsymbol{k}_{mn}|}\tilde{B}_{z,mn} \tag{A.19}$$

$$\tilde{B}_{y,mn} = \frac{\mathrm{i}l_{mn}}{|\boldsymbol{k}_{mn}|}\tilde{B}_{z,mn} \tag{A.20}$$

$$\frac{\mathrm{d}\tilde{B}_{z,mn}}{\mathrm{d}z} = |\boldsymbol{k}_{mn}|\tilde{B}_{z,mn}. \tag{A.21}$$

(these apply to both $\boldsymbol{B}^+$ and $\boldsymbol{B}^-$ at $z = 0$). For the mean field components, with $(m, n) = (0, 0)$, we simply have that $\boldsymbol{B}^+_{00} = \boldsymbol{B}^-_{00} = \boldsymbol{B}_\infty$ on the boundary.

We can also derive the equivalent condition for a potential field *below* the layer (i.e. a potential field in $z > 1$). The same derivation as above can be used; the results are the same except that a minus sign must be inserted on the right-hand side of each of (A.19), (A.20) and (A.21).

Appendix B

Representations for $\boldsymbol{B}$

In this Appendix we consider two different alternative representations for the magnetic field: the vector potential ($\boldsymbol{A}$) and poloidal and toroidal potentials (B_P and B_T). These are useful when programming numerical simulations for MHD, since they provide ways of ensuring that Maxwell's equation $\nabla \cdot \boldsymbol{B} = 0$ is satisfied.

B.1 The vector potential

Since $\nabla \cdot \boldsymbol{B} = 0$, we may write $\boldsymbol{B}$ as the curl of a vector potential $\boldsymbol{A}$, as follows:

$$\boldsymbol{B} = \nabla \wedge \boldsymbol{A}. \tag{B.1}$$

We also know from another of Maxwell's equations that

$$\nabla \wedge \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{B.2}$$

$$= -\frac{\partial \nabla \wedge \boldsymbol{A}}{\partial t}, \tag{B.3}$$

from which

$$\boldsymbol{E} = -\nabla \Phi - \frac{\partial \boldsymbol{A}}{\partial t}. \tag{B.4}$$

Here Φ is the *scalar potential* for the electric field.

We can derive an evolution equation for $\boldsymbol{A}$ by making use of Ohm's law, which gives another expression for $\boldsymbol{E}$:

$$\boldsymbol{E} = -\boldsymbol{u} \wedge \boldsymbol{B} + \eta \nabla \wedge \boldsymbol{B}. \tag{B.5}$$

Equating the two expressions for $\boldsymbol{E}$ and rearranging, we derive the evolution equation

$$\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{u} \wedge \boldsymbol{B} - \eta \nabla \wedge \boldsymbol{B} - \nabla \Phi \tag{B.6}$$

for $\boldsymbol{A}$.

Note that this representation for $\boldsymbol{E}$ and $\boldsymbol{B}$ is not unique. We can make a *gauge transformation*

$$\begin{aligned} \boldsymbol{A} &\rightarrow \boldsymbol{A} + \nabla\chi && \text{(B.7)} \\ \Phi &\rightarrow \Phi - \frac{\partial \chi}{\partial t} && \text{(B.8)} \end{aligned}$$

(for any function χ) without changing the physical fields $\boldsymbol{E}$ and $\boldsymbol{B}$.

We would now like to make a gauge transformation to eliminate the $\nabla\Phi$ term from (B.6), since that will allow us to eliminate Φ completely from the governing equations for our system. For example, this can be done by choosing χ as follows:

$$\chi = \int \Phi dt, \qquad \text{(B.9)}$$

which will set Φ to zero everywhere. Note that this gauge condition (that $\nabla\boldsymbol{\Phi}$ should be identically zero) does not quite specify $\boldsymbol{A}$ and Φ uniquely; for example, χ could be chosen to be a function of position alone (and not time), which would change $\boldsymbol{A}$ but leave Φ (and hence $\nabla\Phi$) unaltered.

B.1.1 Boundary conditions

Using (B.6) (and an appropriate gauge transformation so that $\nabla\Phi \equiv 0$) we can now evolve our system numerically by using $\boldsymbol{A}$ as a variable instead of $\boldsymbol{B}$. We now need to show how boundary conditions on $\boldsymbol{B}$ can be rewritten as boundary conditions on $\boldsymbol{A}$. It turns out that this is readily done for simple boundary conditions (such as a vertical field condition) but is more troublesome for more complex conditions (e.g. potential or tied field conditions).

Vertical field condition

Here we require $B_x = B_y = 0$ on some horizontal surface ($z =$ const). In terms of $\boldsymbol{A}$ we have

$$\begin{aligned} \frac{\partial A_z}{\partial x} &= \frac{\partial A_x}{\partial z}, && \text{(B.10)} \\ \frac{\partial A_z}{\partial y} &= \frac{\partial A_y}{\partial z}. && \text{(B.11)} \end{aligned}$$

However for a simulation we would require three boundary conditions, one for each component of $\boldsymbol{A}$. The third condition will come from the choice of gauge. We claim that it is possible to choose a gauge in which (a) $A_z = 0$ on the boundary (so that the boundary conditions simply become $A_z = \partial A_x/\partial z = \partial A_y/\partial z = 0$) and (b) $\nabla\Phi \equiv 0$ (so that the evolution equation (B.6) is simplified).

This can be demonstrated as follows. First of all we make the transformation given above (equation B.9) to satisfy condition (b). Now we have

$$\frac{\partial \boldsymbol{A}}{\partial t} = -\boldsymbol{E} \tag{B.12}$$

and it may be verified from (B.5) that $E_z = 0$ on the boundary (if $\boldsymbol{B}$ is vertical there). Hence in this gauge A_z will remain constant in time on the boundary (although it may vary in space).

Now we will make a further transformation that sets A_z to zero on the boundary whilst preserving the condition $\nabla\Phi = 0$ everywhere. First let $a(x, y)$ be the value of A_z on the boundary in the old gauge. Now make the gauge transformation with

$$\chi = -za(x, y); \tag{B.13}$$

we have

$$A_z^{\text{new}} = A_z^{\text{old}} + \frac{\partial \chi}{\partial z} \tag{B.14}$$

$$= A_z^{\text{old}} - a(x, y), \tag{B.15}$$

which is zero on the boundary as required. In addition χ does not depend on time so Φ and hence $\nabla\Phi$ are not changed.

This argument can be generalized to two horizontal boundary surfaces instead of one (e.g. using a gauge transformation of the form $\chi = -za(x, y) + z^2 b(x, y)$).

Other boundary conditions

Unfortunately it is difficult to generalize this argument to other types of boundary condition. Take, for example, the tied field boundary condition, which states that the horizontal components of $\boldsymbol{E}$ must be zero on the boundary surface. If we take $\nabla\Phi = 0$, then it is readily seen from (B.4) that the A_x and A_y will be constant (in time) on the boundary, but A_z will vary with time. Therefore, we have boundary conditions on A_x and A_y, but not A_z.

As above, the boundary condition on A_z will come from the gauge condition. However, we are constrained because we have already chosen to use a gauge in which $\nabla\Phi = 0$.

We must derive a boundary condition on A_z which is consistent with this gauge condition. For the vertical field case, this proved straightforward since we could make a gauge transformation to set $A_z = 0$. However, that gauge transformation will not work in this case because A_z is time-dependent. It is difficult to see how to derive a boundary condition for A_z that is consistent with the condition $\nabla\Phi = 0$.

Another option would be to abandon the choice $\nabla\Phi = 0$. For example, if we used the well-known *Coulomb gauge*, in which $\nabla \cdot \boldsymbol{A} = 0$, we would straight away have a boundary condition for $\partial A_z/\partial z$ in terms of derivatives of the (known) functions A_x and A_y. However, in this gauge $\nabla\Phi \neq 0$ in general, and so we have to deal with the extra complexity in the induction equation (B.6).

Note that in two dimensions, the problems disappear. In this case, we usually have $\boldsymbol{A} = A(x, z)\boldsymbol{e}_y$, and $\Phi \equiv 0$, and it can be seen that the only necessary boundary condition is that A is constant (in time) along the boundary.

B.2 Poloidal/toroidal decomposition

In view of these difficulties with boundary conditions, it seems preferable to use a different approach, in which the gauge-related ambiguities are removed. This can be done by decomposing $\boldsymbol{B}$ into separate poloidal and toroidal potentials, as follows:

$$\boldsymbol{B} = \nabla \wedge (B_T \boldsymbol{e}_z) + \nabla \wedge (\nabla \wedge (B_P \boldsymbol{e}_z)). \tag{B.16}$$

The components of $\boldsymbol{B}$ are now represented as

$$\boldsymbol{B} = \begin{pmatrix} \partial B_T/\partial y + \partial^2 B_P/\partial x \partial z \\ -\partial B_T/\partial x + \partial^2 B_P/\partial y \partial z \\ -\nabla_H^2 B_P \end{pmatrix} \tag{B.17}$$

where $\nabla_H^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$.

The induction equation for $\boldsymbol{B}$ is

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}. \tag{B.18}$$

We take the scalar product of this equation with $\boldsymbol{e}_z$ to obtain an equation for B_P:

$$\frac{\partial B_P}{\partial t} = \eta \nabla^2 B_P - \nabla_H^{-2}[\boldsymbol{e}_z \cdot \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{B})] \tag{B.19}$$

(note that the ∇_H^{-2} operator can easily be computed in Fourier space). Furthermore, taking the curl of the induction equation and then forming the scalar product with $\boldsymbol{e}_z$

produces the following equation for B_T:

$$\frac{\partial B_T}{\partial t} = \eta \nabla^2 B_T - \nabla_H^{-2} [\boldsymbol{e}_z \cdot \nabla \wedge \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{B})]. \tag{B.20}$$

Let

$$\boldsymbol{\mathcal{E}} = \boldsymbol{u} \wedge \boldsymbol{B} = \begin{pmatrix} u_y B_z - u_z B_y \\ u_z B_x - u_x B_z \\ u_x B_y - u_y B_x \end{pmatrix}. \tag{B.21}$$

Then

$$\boldsymbol{e}_z \cdot \nabla \wedge \boldsymbol{\mathcal{E}} = \frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y} \tag{B.22}$$

and

$$\boldsymbol{e}_z \cdot \nabla \wedge \nabla \wedge \boldsymbol{\mathcal{E}} = \frac{\partial^2 \mathcal{E}_x}{\partial x \partial z} + \frac{\partial^2 \mathcal{E}_y}{\partial y \partial z} - \nabla_H^2 \mathcal{E}_z \tag{B.23}$$

so

$$\frac{\partial B_T}{\partial t} = \eta \nabla^2 B_T + \mathcal{E}_z - \nabla_H^{-2} \left(\frac{\partial^2 \mathcal{E}_x}{\partial x \partial z} + \frac{\partial^2 \mathcal{E}_y}{\partial y \partial z} \right) \tag{B.24}$$

$$\frac{\partial B_P}{\partial t} = \eta \nabla^2 B_P + \nabla_H^{-2} \left(\frac{\partial \mathcal{E}_x}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial x} \right). \tag{B.25}$$

B.2.1 A note about periodicity

Simulations are usually run with periodic boundary conditions in x and y; however, if we require B_T and B_P to be periodic, then we find that we cannot represent a uniform magnetic field. The solution is to treat the mean part of the field separately:

$$\boldsymbol{B} = \nabla \wedge (B_T \boldsymbol{e}_z) + \nabla \wedge (\nabla \wedge (B_P \boldsymbol{e}_z)) + \bar{\boldsymbol{B}} \tag{B.26}$$

where B_T and B_P are periodic and $\bar{\boldsymbol{B}}$ is the mean field. We can assume without loss of generality that the horizontal means of both B_T and B_P are zero. $\bar{B}_z$ is constant because vertical flux is conserved, while $\bar{B}_x$ and $\bar{B}_y$ will be functions of z and t.

This change does not explicitly affect the evolution equations for B_T and B_P. However, a new evolution equation for $\bar{\boldsymbol{B}}$ must be added. This can be found by averaging the induction equation:

$$\frac{\partial \bar{\boldsymbol{B}}}{\partial t} = \nabla \wedge (\overline{\boldsymbol{u} \wedge \boldsymbol{B}}) + \eta \nabla^2 \bar{\boldsymbol{B}} \tag{B.27}$$

(where the bar represents an average over x and y).

B.2.2 Boundary conditions

Vertical field

Imposing a vertical field boundary condition is straightforward, since we require only $B_x = B_y = 0$, which gives

$$B_T = \frac{\partial B_P}{\partial z} = 0; \quad \bar{B}_x = \bar{B}_y = 0 \tag{B.28}$$

at the boundaries.

Potential field

For a potential field above the layer we require (in Fourier space)

$$B_x = \frac{\mathrm{i}k}{|\boldsymbol{k}|} B_z \tag{B.29}$$

$$B_y = \frac{\mathrm{i}l}{|\boldsymbol{k}|} B_z \tag{B.30}$$

$$\frac{\partial B_z}{\partial z} = |\boldsymbol{k}| B_z. \tag{B.31}$$

The last of these implies

$$\frac{\partial B_P}{\partial z} = |\boldsymbol{k}| B_P \tag{B.32}$$

and then (B.29) and (B.30) together imply

$$B_T = 0. \tag{B.33}$$

The mean field on the boundaries is simply set to the value of the potential field at infinity.

Tied field

For tied field lines we use the fact that

$$E_x = E_y = 0 \tag{B.34}$$

on the boundary. Substituting for $\boldsymbol{E}$ from (B.5), we obtain

$$-\mathcal{E}_x + \eta \frac{\partial^2 B_T}{\partial x \partial z} - \eta \frac{\partial \nabla^2 B_P}{\partial y} = 0 \tag{B.35}$$

$$-\mathcal{E}_y + \eta \frac{\partial^2 B_T}{\partial y \partial z} + \eta \frac{\partial \nabla^2 B_P}{\partial x} = 0 \tag{B.36}$$

where $\boldsymbol{\mathcal{E}}$ is as defined in (B.21). Differentiating (B.35) w.r.t. y and (B.36) w.r.t. x, and subtracting the results, gives

$$-\frac{\partial \mathcal{E}_x}{\partial y} + \frac{\partial \mathcal{E}_y}{\partial x} - \eta \nabla^2 \nabla_H^2 B_P = 0 \tag{B.37}$$

which, from the induction equation, is equivalent to

$$\frac{\partial B_P}{\partial t} = 0. \tag{B.38}$$

(In other words this is a Dirichlet condition with B_P specified on the boundary.) We can derive a condition on B_T by differentiating (B.35) w.r.t. x and (B.36) w.r.t. y and adding the results:

$$-\frac{\partial \mathcal{E}_x}{\partial x} - \frac{\partial \mathcal{E}_y}{\partial y} + \eta \frac{\partial \nabla_H^2 B_T}{\partial z} = 0 \tag{B.39}$$

which can be rearranged to give a condition on $\partial B_T / \partial z$:

$$\frac{\partial B_T}{\partial z} = \frac{1}{\eta} \nabla_H^{-2} \left(\frac{\partial \mathcal{E}_x}{\partial x} + \frac{\partial \mathcal{E}_y}{\partial y} \right) \tag{B.40}$$

The boundary conditions for the mean field must be found separately. From substituting $\bar{E}_x = \bar{E}_y = 0$ into (B.5) we find

$$\frac{\partial \bar{B}_x}{\partial z} = -\frac{1}{\eta} \overline{u_x B_z} \tag{B.41}$$

$$\frac{\partial \bar{B}_y}{\partial z} = -\frac{1}{\eta} \overline{u_y B_z} \tag{B.42}$$

(using $u_z = 0$ on the boundary).

Appendix C

Weakly nonlinear derivations

In this Appendix we will show how the various solution branches to our weakly nonlinear models, including the oscillatory hexagonal and the rhombic models, were found, indicating how the existence and stability results were calculated.

C.1 The formula for A_c in the steady hexagonal model

Here we show how to derive equation (3.15) from chapter 3. This formula applied to the steady hexagonal model with $\theta = 30°$, so that $A_1 > A_2 = A_3 = 1$.

We derive this equation by using the results of Malomed et al. (1994), and in particular their Figure 3. Their γ corresponds to our $\mu_{2\mathrm{r}} = \mu_{3\mathrm{r}}$, and their γ_3 corresponds to our $\mu_{1\mathrm{r}}$. When $r = 0$ we are essentially following the path

$$\begin{aligned} \gamma_3 &= A_1 \phi^2 \\ \gamma &= \phi^2 \end{aligned}$$

which is the straight line

$$\gamma = (1/A_1)\gamma_3. \tag{C.1}$$

Their curve FHK represents the pitchfork bifurcation at which R_1 rolls become stable. (This curve has the equation $\gamma = \beta\gamma_3 - \sqrt{\gamma_3}$.) If the line (C.1) crosses this curve to the left of the point H on their diagram, then the pitchfork is supercritical; if it crosses to the right of H, the pitchfork is subcritical. The point H corresponds to

$$\gamma_3 = \gamma_3^+ \equiv \frac{(2\beta + \sqrt{2\beta + 2})^2}{4(1 + \beta - 2\beta^2)^2} \tag{C.2}$$

while the 'crossing point' corresponds to

$$\frac{1}{A_1}\gamma_3 = \beta\gamma_3 - \sqrt{\gamma_3} \tag{C.3}$$

which has the non-trivial solution

$$\gamma_3 = \frac{1}{\left(\frac{1}{A_1} - \beta\right)^2}. \tag{C.4}$$

A supercritical bifurcation will therefore occur if and only if

$$\frac{1}{\left(\frac{1}{A_1} - \beta\right)^2} < \frac{(2\beta + \sqrt{2\beta + 2})^2}{4(1 + \beta - 2\beta^2)^2}. \tag{C.5}$$

Since $\beta > 1$ and $1/A_1 < 1$ by assumption, we have that $1/A_1 - \beta < 0$, while $2\beta + \sqrt{2\beta + 2}$ is certainly positive. It can also be shown that $1 + \beta - 2\beta^2$ is always negative for $\beta > 1$. Therefore the inequality (C.5) is equivalent to

$$\frac{1}{\beta - 1/A_1} < \frac{2\beta + \sqrt{2\beta + 2}}{-2(1 + \beta - 2\beta^2)} \tag{C.6}$$

which can be rearranged to give

$$\frac{1}{A_1} < \beta + \frac{2(1 + \beta - 2\beta^2)}{2\beta + \sqrt{2\beta + 2}}. \tag{C.7}$$

If the right-hand side is positive, then this corresponds to a supercritical bifurcation for $A_1 > A_\mathrm{c}$, and a subcritical bifurcation for $A_1 < A_\mathrm{c}$, where A_c is the reciprocal of the right-hand side of (C.7). If A_c turns out to be negative, however, then the bifurcation is always *subcritical*.

C.2 Solving the weakly nonlinear amplitude equations: Some predefined equation systems

We now turn to the derivations of the various solution branches to the amplitude equations from Chapter 3, and in particular the oscillatory models (on both the rhombic and the hexagonal lattices).

There are a few systems of equations that will appear many times during the following analysis. We will refer to these as 'system 1' through to 'system 6'. It will save time if we present the solutions (and stability criteria) for these common equation systems now, rather than repeating the results later. The reader may prefer to skip ahead to the next section, and refer back as needed to the equation systems defined here.

C.2.1 System 1

This is a linear system given by

$$\dot{x} = Ax + Be^{i\omega t}y \tag{C.8}$$

$$\dot{y} = Be^{-i\omega t}x + Cy \tag{C.9}$$

where A and B are complex and ω is real. We can solve it by substituting $z = e^{i\omega t}y$:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A & B \\ B & C + i\omega \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \tag{C.10}$$

If the eigenvalues of the above matrix both have negative real parts, then the solutions are exponentially decaying; otherwise they are exponentially growing.

C.2.2 System 2

System 2 will be the normal form equations for the Hopf bifurcation problem with O(2) symmetry, as follows:

$$\dot{x} = Ax + C|x|^2x + D|y|^2x \tag{C.11}$$

$$\dot{y} = By + C|y|^2y + D|x|^2y \tag{C.12}$$

This is a standard problem. We are interested in the solution with x and y both nonzero, for which we may decompose the system into amplitude and phase equations by substituting $x = Re^{i\theta}$, $y = Se^{i\phi}$; this leads to the following solution for R, S, θ and ϕ:

$$|x|^2 = \frac{B_r D_r - A_r C_r}{C_r^2 - D_r^2} \tag{C.13}$$

$$|y|^2 = \frac{A_r D_r - B_r C_r}{C_r^2 - D_r^2} \tag{C.14}$$

$$d/dt(\arg x) = A_i + C_i|x|^2 + D_i|y|^2 \tag{C.15}$$

$$d/dt(\arg y) = B_i + C_i|y|^2 + D_i|x|^2. \tag{C.16}$$

The condition for existence of these solutions is that $|x|^2$ and $|y|^2$ both be positive; the condition for stability is that both $C_r < 0$ and $|C_r| > |D_r|$.

C.2.3 System 3

This is a linear system defined as follows:

$$\dot{x} = Ax + Be^{i\omega t}\bar{y} \tag{C.17}$$

$$\dot{y} = Cy + Be^{i\omega t}\bar{x} \tag{C.18}$$

To solve, substitute $z = e^{i\omega t}\bar{y}$:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A & B \\ \bar{B} & \bar{C} + i\omega \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \tag{C.19}$$

The equations have exponentially decaying solutions if and only if the eigenvalues of this matrix both have negative real parts.

C.2.4 System 4

System 4 is the following:

$$\dot{x} = Ax + \alpha_3 e^{i(-\omega_1+\omega_2)t}y + \alpha_2 e^{i(-\omega_1+\omega_3)t}z \tag{C.20}$$

$$\dot{y} = By + \alpha_1 e^{i(-\omega_2+\omega_3)t}z + \alpha_3 e^{i(\omega_1-\omega_2)t}x \tag{C.21}$$

$$\dot{z} = Cz + \alpha_2 e^{i(\omega_1-\omega_3)t}x + \alpha_1 e^{i(\omega_2-\omega_3)t}y \tag{C.22}$$

where A, B and C are complex and ω_1, ω_2 and ω_3 are real.

To solve, substitute $X = e^{i\omega_1 t}x$, $Y = e^{i\omega_2 t}y$, $Z = e^{i\omega_3 t}z$, to obtain

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{pmatrix} i\omega_1 + A & \alpha_3 & \alpha_2 \\ \alpha_3 & i\omega_2 + B & \alpha_1 \\ \alpha_2 & \alpha_1 & i\omega_3 + C \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}. \tag{C.23}$$

The three eigenvalues of this matrix must have negative real parts for stability.

C.2.5 System 5

System 5 is the following nonlinear system:

$$\dot{x} = Ax + D|x|^2 x + E(|y|^2 + |z|^2)x \tag{C.24}$$

$$\dot{y} = By + D|y|^2 y + E(|x|^2 + |z|^2)y \tag{C.25}$$

$$\dot{z} = Cz + D|z|^2 z + E(|x|^2 + |y|^2)z \tag{C.26}$$

As usual, we are interested in solutions with constant non-zero amplitudes $|x|$, $|y|$ and $|z|$, but time-varying phases. These will be as follows:

$$|x|^2 = \frac{E_\mathrm{r}(B_\mathrm{r}+C_\mathrm{r})-(D_\mathrm{r}+E_\mathrm{r})A_\mathrm{r}}{(D_\mathrm{r}-E_\mathrm{r})(D_\mathrm{r}+2E_\mathrm{r})} \tag{C.27}$$

$$|y|^2 = \frac{E_\mathrm{r}(A_\mathrm{r}+C_\mathrm{r})-(D_\mathrm{r}+E_\mathrm{r})B_\mathrm{r}}{(D_\mathrm{r}-E_\mathrm{r})(D_\mathrm{r}+2E_\mathrm{r})} \tag{C.28}$$

$$|z|^2 = \frac{E_\mathrm{r}(A_\mathrm{r}+B_\mathrm{r})-(D_\mathrm{r}+E_\mathrm{r})C_\mathrm{r}}{(D_\mathrm{r}-E_\mathrm{r})(D_\mathrm{r}+2E_\mathrm{r})} \tag{C.29}$$

with

$$\frac{\mathrm{d}}{\mathrm{d}t}(\arg x) = A_\mathrm{i}+D_\mathrm{i}|x|^2+E_\mathrm{i}(|y|^2+|z|^2) \tag{C.30}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\arg y) = B_\mathrm{i}+D_\mathrm{i}|y|^2+E_\mathrm{i}(|x|^2+|z|^2) \tag{C.31}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\arg z) = C_\mathrm{i}+D_\mathrm{i}|z|^2+E_\mathrm{i}(|x|^2+|y|^2) \tag{C.32}$$

For stability, note that the amplitude and phase equations decouple, so we only need to consider stability for the amplitude equations. The Jacobian of the three amplitude equations is as follows:

$$\begin{pmatrix} A_\mathrm{r}+E_\mathrm{r}(|y|^2+|z|^2)+3D_\mathrm{r}|x|^2 & 2E_\mathrm{r}|x||y| & 2E_\mathrm{r}|x||z| \\ 2E_\mathrm{r}|x||y| & B_\mathrm{r}+E_\mathrm{r}(|x|^2+|z|^2)+3D_\mathrm{r}|y|^2 & 2E_\mathrm{r}|y||z| \\ 2E_\mathrm{r}|x||z| & 2E_\mathrm{r}|y||z| & C_\mathrm{r}+E_\mathrm{r}(|x|^2+|y|^2)+3D_\mathrm{r}|z|^2 \end{pmatrix}. \tag{C.33}$$

The solutions are stable if the three eigenvalues of this matrix have negative real parts.

C.2.6 System 6

This is a generalization of system 5, defined as follows:

$$\dot{x} = Ax+D|x|^2x+E|y|^2x+F|z|^2x \tag{C.34}$$

$$\dot{y} = By+D|y|^2y+E|x|^2y+F|z|^2y \tag{C.35}$$

$$\dot{z} = Cz+D|z|^2z+F(|x|^2+|y|^2)z \tag{C.36}$$

To solve this, we can set $\mathrm{Re}(\dot{x}/x) = \mathrm{Re}(\dot{y}/y) = \mathrm{Re}(\dot{z}/z) = 0$, which results in a matrix equation which can be solved numerically to obtain the three amplitudes. The solutions will be stable if the eigenvalues of the following matrix both have negative real parts:

$$\begin{pmatrix} A_\mathrm{r}+3D_\mathrm{r}|x|^2+E_\mathrm{r}|y|^2+F_\mathrm{r}|z|^2 & 2E_\mathrm{r}|x||y| & 2F_\mathrm{r}|x||z| \\ 2E_\mathrm{r}|x||y| & B_\mathrm{r}+E_\mathrm{r}|x|^2+3D_\mathrm{r}|y|^2+F_\mathrm{r}|z|^2 & 2F_\mathrm{r}|y||z| \\ 2F_\mathrm{r}|x||z| & 2F_\mathrm{r}|y||z| & C_\mathrm{r}+3D_\mathrm{r}|z|^2+F_\mathrm{r}(|x|^2+|y|^2) \end{pmatrix}. \tag{C.37}$$

C.3 Rhombic model

For the oscillatory model of section 3.5, the equations to be solved are:

$$\dot{z}_1 = \mu z_1 + (a|z_1|^2 + b|w_2|^2 + b|w_1|^2 + 2a|z_2|^2)z_1 + bw_2z_2\bar{w}_1 \quad \text{(C.38)}$$

$$\dot{z}_2 = \mu z_2 + (a|z_2|^2 + b|w_1|^2 + b|w_2|^2 + 2a|z_1|^2)z_2 + bw_1z_1\bar{w}_2 \quad \text{(C.39)}$$

$$\dot{w}_1 = \mu' w_1 + (a|w_1|^2 + b|z_2|^2 + b|z_1|^2 + 2a|w_2|^2)w_1 + bw_2z_2\bar{z}_1 \quad \text{(C.40)}$$

$$\dot{w}_2 = \mu' w_2 + (a|w_2|^2 + b|z_1|^2 + b|z_2|^2 + 2a|w_1|^2)w_2 + bw_1z_1\bar{z}_2 \quad \text{(C.41)}$$

We look for solutions in which the amplitudes are constant and the phases are periodic functions of time. We do this by splitting into different cases depending on how many of the amplitudes are non-zero.

C.3.1 One non-zero amplitude

If three of the four complex amplitudes are zero we get travelling rolls. We will take z_1 to be non-zero, to obtain TRo^{R} (TRo^{R} with z_2 non-zero are equivalent, and TRo^{L}, with either w_1 or w_2 non-zero, can be found analogously). The equation for z_1 is

$$\dot{z}_1 = \mu z_1 + a|z_1|^2 z_1 \quad \text{(C.42)}$$

This has the solution

$$|z_1|^2 = -\mu_{\text{r}}/a_{\text{r}} \quad \text{(C.43)}$$

$$\frac{\text{d}}{\text{d}t}(\arg z_1) = \mu_{\text{i}} + a_{\text{i}}|z_1|^2 \quad \text{(C.44)}$$

The solutions are stable to perturbations in z_1 if $a_{\text{r}} < 0$.

The perturbations to the other variables decouple:

$$\delta\dot{z}_2 = (\mu + 2a|z_1|^2)\delta z_2 \quad \text{(C.45)}$$

$$\delta\dot{w}_1 = (\mu' + b|z_1|^2)\delta w_1 \quad \text{(C.46)}$$

$$\delta\dot{w}_2 = (\mu' + b|z_1|^2)\delta w_2 \quad \text{(C.47)}$$

For stability, these must all have exponentially decaying solutions, which is true iff $\mu'_{\text{r}}/\mu_{\text{r}} < b_r/a_r$.

C.3.2 Two non-zero amplitudes

Standing rolls (SRo)

For example, z_1 and z_2 non-zero. The equations for these variables are

$$\dot{z}_1 = \mu z_1 + (a|z_1|^2 + b|w_1|^2)z_1 \tag{C.48}$$

$$\dot{w}_1 = \mu' w_1 + (a|w_1|^2 + b|z_1|^2)w_1 \tag{C.49}$$

This corresponds to our 'system 2' (see page 217), with $A = \mu$, $B = \mu'$, $C = a$, $D = b$, $x = z_1$ and $y = w_1$. The stability conditions are: $a_\mathrm{r} < 0$ and $|a_\mathrm{r}| > |b_\mathrm{r}|$.

Perturbations to the other variables give

$$\delta\dot{z}_2 = (\mu + b|w_1|^2 + 2a|z_1|^2)\delta z_2 + bw_1 z_1 \delta\bar{w}_2 \tag{C.50}$$

$$\delta\dot{w}_2 = (\mu' + b|z_1|^2 + 2a|w_1|^2)\delta w_2 + bz_1 w_1 \delta\bar{z}_2 \tag{C.51}$$

This corresponds to system 3 (defined on page 218), with $A = \mu + b|w_1|^2 + 2a|z_1|^2$, $B = b|w_1||z_1|$, $C = \mu' + b|z_1^2| + 2a|w_1|^2$, $\omega = \mathrm{d}/\mathrm{d}t(\arg z_1) + \mathrm{d}/\mathrm{d}t(\arg w_1) = \mu_\mathrm{i} + \mu'_\mathrm{i} + (a_\mathrm{i} + b_\mathrm{i})(|z_1|^2 + |w_1|^2)$. We have stability when the eigenvalues of the following matrix both have negative real parts:

$$M = \begin{pmatrix} \mu + b|w_1|^2 + 2a|z_1|^2 & b|w_1||z_1| \\ \bar{b}|w_1||z_1| & \bar{\mu}' + \bar{b}|z_1|^2 + 2\bar{a}|w_1|^2 + \mathrm{i}\omega \end{pmatrix}. \tag{C.52}$$

We can simplify this: by eliminating μ_r and μ'_r in favour of $|z_1|^2$ and $|w_1|^2$, we obtain

$$M = \begin{pmatrix} a|z_1|^2 + \mathrm{i}(\mu_\mathrm{i} + b_\mathrm{i}|w_1|^2 + a_\mathrm{i}|z_1|^2) & b|w_1||z_1| \\ \bar{b}|w_1||z_1| & \bar{a}|w_1|^2 + \mathrm{i}(\mu_\mathrm{i} + a_\mathrm{i}|z_1|^2 + b_\mathrm{i}|w_1|^2) \end{pmatrix}. \tag{C.53}$$

We can now add $-\mathrm{i}(\mu_\mathrm{i} + b_\mathrm{i}|w_1|^2 + a_\mathrm{i}|z_1|^2)$ times the identity matrix to M; this does not affect the real parts of its eigenvalues. The resulting matrix is

$$M' = \begin{pmatrix} a|z_1|^2 & b|w_1||z_1| \\ \bar{b}|w_1||z_1| & \bar{a}|w_1|^2 \end{pmatrix}. \tag{C.54}$$

The trace of this matrix is $a(|z_1|^2 + |w_1|^2)$ which has negative real part (since $a_\mathrm{r} < 0$ is already a stability condition, see above). The determinant is $(|a|^2 - |b|^2)\ |w_1|^2|z_1|^2$, which is negative and has the sign of $|a|^2 - |b|^2$. From this information we see that the two eigenvalues of M' (and therefore also M) either both have negative real parts, if $|a|^2 > |b|^2$, or both have positive real parts, if $|a|^2 < |b|^2$.

Overall, therefore, SRo are stable if $a_\mathrm{r} < 0$, $|a_\mathrm{r}| > |b_\mathrm{r}|$ and $|a| > |b|$.

Leftward- and rightward-travelling rectangles ($\mathrm{TRe}^{\mathrm{L}}$ and $\mathrm{TRe}^{\mathrm{R}}$)

We look at $\mathrm{TRe}^{\mathrm{R}}$ with z_1 and z_2 non-zero. The equations for z_1 and z_2 are

$$\dot{z}_1 = \mu z_1 + (a|z_1|^2 + 2a|z_2|^2)z_1 \tag{C.55}$$

$$\dot{z}_2 = \mu z_2 + (a|z_2|^2 + 2a|z_1|^2)z_2 \tag{C.56}$$

This corresponds to system 2 (page 217), with $C = a$ and $D = 2a$. Notice that the stability condition for system 2 is not satisfied with these values for C and D. (The same applies for $\mathrm{TRe}^{\mathrm{R}}$.) Therefore left- or right-travelling rectangles can never be stable.

Perpendicular travelling rectangles ($\mathrm{TRe}^{\perp}$)

For example, z_1 and w_2 non-zero. The equations are

$$\dot{z}_1 = \mu z_1 + (a|z_1|^2 + b|w_2|^2)z_1 \tag{C.57}$$

$$\dot{w}_2 = \mu' w_2 + (a|w_2|^2 + b|z_1|^2)w_2. \tag{C.58}$$

This corresponds to system 2 (page 217) with $A = \mu$, $B = \mu'$, $C = a$, $D = b$, $x = z_1$ and $y = w_2$. The stability condition is $a_{\mathrm{r}} < 0$ and $|a_{\mathrm{r}}| > |b_{\mathrm{r}}|$.

The perturbations to z_2 and w_1 satisfy

$$\delta\dot{z}_2 = (\mu + b|w_2|^2 + 2a|z_1|^2)\delta z_2 + b z_1 \bar{w}_2 \delta w_1 \tag{C.59}$$

$$\delta\dot{w}_1 = (\mu' + b|z_1|^2 + 2a|w_2|^2)\delta w_1 + b w_2 \bar{z}_1 \delta z_2. \tag{C.60}$$

This corresponds to system 1 (page 217), with $A = \mu + b|w_2|^2 + 2a|z_1|^2$, $B = b|z_1||w_2|$, $C = \mu' + b|z_1|^2 + 2a|w_2|^2$, $\omega = \mu_{\mathrm{i}} - \mu'_{\mathrm{i}} + (a_{\mathrm{i}} - b_{\mathrm{i}})(|z_1|^2 - |w_2|^2)$. We use similar methods as in section C.3.2: we take the stability matrix from system 1, eliminate μ and μ', and add an imaginary multiple of the identity matrix, eventually obtaining the following matrix:

$$M = \begin{pmatrix} a|z_1|^2 & b|z_1||w_2| \\ b|z_1||w_2| & a|w_2|^2 \end{pmatrix}. \tag{C.61}$$

We now show that if $a_{\mathrm{r}} < 0$ and $|a_{\mathrm{r}}| > |b_{\mathrm{r}}|$, then both eigenvalues of M always have negative real parts. We first of all note that without loss of generality, we can rescale such that $a_{\mathrm{r}} = -1$ and $|z_1| = 1$. The eigenvalues λ of M are now given by

$$(a - \lambda)(aY - \lambda) - b^2 Y = 0 \tag{C.62}$$

where Y is a shorthand for $|w_2|^2$. We proceed by looking for bifurcation points where $\lambda_\mathrm{r} = 0$, i.e. where $\lambda = \mathrm{i}\omega$ (ω real). Equation (C.62) becomes

$$(-1 + \mathrm{i}(a_\mathrm{i} - \omega))(-Y + \mathrm{i}(a_\mathrm{i}Y - \omega)) - (b_\mathrm{r}^2 - b_\mathrm{i}^2)Y - 2\mathrm{i}b_\mathrm{r}b_\mathrm{i}Y = 0, \tag{C.63}$$

which may be split into real and imaginary parts, yielding the following two equations:

$$Y - (a_\mathrm{i} - \omega)(a_\mathrm{i}Y - \omega) - (b_\mathrm{r}^2 - b_\mathrm{i}^2)Y = 0 \tag{C.64}$$

$$-2a_\mathrm{i}Y + \omega Y + \omega - 2b_\mathrm{r}b_\mathrm{i}Y = 0. \tag{C.65}$$

Equation (C.65) gives an expression for ω, which can then be substituted into (C.64) to obtain (after some simplification):

$$(1 - b_\mathrm{r}^2 + a_\mathrm{i}^2 + b_\mathrm{i}^2) + 2b_\mathrm{r}b_\mathrm{i}a_\mathrm{i}\left(1 - 4\frac{Y}{(1+Y)^2}\right) - 4(b_\mathrm{r}^2 b_\mathrm{i}^2 + a_\mathrm{i}^2)\frac{Y}{(1+Y)^2} = 0. \tag{C.66}$$

The left hand side is minimized with respect to Y (under the constraint $Y > 0$) when $Y = 1$, when it takes the value $(1-b_\mathrm{r}^2)(1+b_\mathrm{i}^2)$. Thus, if $|b_\mathrm{r}| < 1$, then equation (C.66) can never be satisfied; therefore there are no bifurcation points in the region $|b_\mathrm{r}| < 1$. Since the eigenvalues of M can readily be shown to both be negative at some point within this region (for example if a and b are both real), then it follows, by continuity, that they must both be negative throughout the whole region. (If this were not true then there would have to be a bifurcation point somewhere at which one of the eigenvalues could change sign.)

Therefore, we conclude that $\mathrm{TRe}^\perp$ are stable iff $a_\mathrm{r} < 0$ and $|a_\mathrm{r}| > |b_\mathrm{r}|$.

C.3.3 Three nonzero amplitudes

This is not possible – for example, if only w_2 is zero, then an inspection of the $\dot{w}_2$ equation shows that $\mathrm{Re}\,\dot{w}_2$ will not be zero (in general), so this will not be a solution (of the type we are looking for here).

C.3.4 Four nonzero amplitudes

If all four amplitudes are nonzero we can write the equations in terms of amplitudes and phases, by setting $z_j = R_j \exp(\mathrm{i}\theta_j)$, $w_j = S_j \exp(\mathrm{i}\phi_j)$. This yields the following set of equations:

$$\begin{aligned}
\dot{R}_1 &= \mu_r R_1 + (a_r R_1^2 + b_r S_2^2 + b_r S_1^2 + 2a_r R_2^2)R_1 + |b|\cos(\psi_0 - \psi)S_2 S_1 R_2 \quad \text{(C.67)}\\
\dot{R}_2 &= \mu_r' S_2 + (a_r S_2^2 + b_r R_1^2 + b_r R_2^2 + 2a_r S_1^2)S_2 + |b|\cos(\psi_0 + \psi)R_1 S_1 R_2 \quad \text{(C.68)}\\
\dot{R}_3 &= \mu_r' S_1 + (a_r S_1^2 + b_r R_2^2 + b_r R_1^2 + 2a_r S_2^2)S_1 + |b|\cos(\psi_0 - \psi)R_1 S_2 R_2 \quad \text{(C.69)}\\
\dot{R}_4 &= \mu_r R_2 + (a_r R_2^2 + b_r S_1^2 + b_r S_2^2 + 2a_r R_1^2)R_2 + |b|\cos(\psi_0 + \psi)R_1 S_2 S_1 \quad \text{(C.70)}\\
\dot{\psi} &= a_i(-R_1^2 + S_2^2 - S_1^2 + R_2^2) + \left(\frac{S_2 S_1 R_2}{R_1} + \frac{R_1 S_2 R_2}{S_1}\right)|b|\sin(\psi_0 - \psi)\\
&\quad - \left(\frac{R_1 S_1 R_2}{S_2} + \frac{R_1 S_2 S_1}{R_2}\right)|b|\sin(\psi_0 + \psi) \quad \text{(C.71)}
\end{aligned}$$

where $\psi = \theta_1 - \theta_2 + \phi_1 - \phi_2$.

The equations are now in a form suitable for investigation with AUTO.

C.4 Oscillatory hexagonal model

For the oscillatory hexagonal model, we wish to solve the following equations:

$$\dot{z}_1 = [\mu_1 + a|z_1|^2 + b|w_1|^2 + c(|z_2|^2 + |z_3|^2) + d(|w_2|^2 + |w_3|^2)]z_1 + f(z_2 w_2 + z_3 w_3)\bar{w}_1 \quad \text{(C.72)}$$

$$\dot{z}_2 = [\mu_2 + a|z_2|^2 + b|w_2|^2 + c(|z_3|^2 + |z_1|^2) + d(|w_3|^2 + |w_1|^2)]z_2 + f(z_3 w_3 + z_1 w_1)\bar{w}_2 \quad \text{(C.73)}$$

$$\dot{z}_3 = [\mu_3 + a|z_3|^2 + b|w_3|^2 + c(|z_1|^2 + |z_2|^2) + d(|w_1|^2 + |w_2|^2)]z_3 + f(z_1 w_1 + z_2 w_2)\bar{w}_3 \quad \text{(C.74)}$$

$$\dot{w}_1 = [\mu_1' + a|w_1|^2 + b|z_1|^2 + c(|w_2|^2 + |w_3|^2) + d(|z_2|^2 + |z_3|^2)]w_1 + f(z_2 w_2 + z_3 w_3)\bar{z}_1 \quad \text{(C.75)}$$

$$\dot{w}_2 = [\mu_2' + a|w_2|^2 + b|z_2|^2 + c(|w_3|^2 + |w_1|^2) + d(|z_3|^2 + |z_1|^2)]w_2 + f(z_3 w_3 + z_1 w_1)\bar{z}_2 \quad \text{(C.76)}$$

$$\dot{w}_3 = [\mu_3' + a|w_3|^2 + b|z_3|^2 + c(|w_1|^2 + |w_2|^2) + d(|z_1|^2 + |z_2|^2)]w_3 + f(z_1 w_1 + z_2 w_2)\bar{z}_3 \quad \text{(C.77)}$$

For simplicity we assume that the real parts of the coefficients a–d and f are all negative. (This ensures that all solutions bifurcate supercritically at onset.)

To solve these, we consider cases in which a number of the z_j and w_j are set equal to zero. We start with the case where only one of these amplitudes is non-zero (which gives travelling rolls), and move up to the situation where all six of them are non-zero. As before, our approach will be to look for solutions in which the amplitudes are constant, although the phases may vary with time.

In the following sections we will calculate the solutions and their stability by making use of the various equation systems already examined in section C.2. The findings will be presented in a number of tables, each of which will show the equation systems to be solved together with the relevant parameters and information on how the results are to be interpreted. (We have also written a FORTRAN program to automate these existence and stability calculations.)

C.4.1 One non-zero amplitude

This corresponds to travelling rolls. We investigate 'z_1 rolls' as an example; the others can be obtained by cyclic permutations.

The amplitude z_1 satisfies

$$\dot{z}_1 = \mu_1 z_1 + a|z_1|^2 z_1. \tag{C.78}$$

This has the following solution in which $|z_1|$ is constant:

$$|z_1|^2 = -\mu_{1\mathrm{r}}/a_\mathrm{r} \tag{C.79}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\arg z_1) = \mu_{1\mathrm{i}} + a_\mathrm{i}|z_1|^2. \tag{C.80}$$

To investigate stability, we now introduce perturbations $z_1 \rightarrow z_1 + \delta z_1$, $z_2 \rightarrow \delta z_2$, etc., and discard terms which are quadratic or higher in the perturbations. After doing this, the equation for $\dot{z}_1$ decouples from the other five equations.

The solutions of the $\dot{z}_1$ equation will be stable if they bifurcate for positive $\mu_{1\mathrm{r}}$ (i.e. $a_\mathrm{r} < 0$), and unstable otherwise.

The other five equations, when linearized, yield the following:

$$\delta\dot{z}_2 = (\mu_2 + c|z_1|^2)\delta z_2 \tag{C.81}$$

$$\delta\dot{z}_3 = (\mu_3 + c|z_1|^2)\delta z_3 \tag{C.82}$$

$$\delta\dot{w}_1 = (\mu_1' + b|z_1|^2)\delta w_1 \tag{C.83}$$

$$\delta\dot{w}_2 = (\mu_2' + d|z_1|^2)\delta w_2 \tag{C.84}$$

$$\delta\dot{w}_3 = (\mu_3' + d|z_1|^2)\delta w_3 \tag{C.85}$$

For stability we require the solutions of these equations to decay, which is the case when all of the following conditions are met:

$$\frac{\mu_{2\mathrm{r}}}{\mu_{1\mathrm{r}}} < \frac{c_\mathrm{r}}{a_\mathrm{r}}, \quad \frac{\mu_{3\mathrm{r}}}{\mu_{1\mathrm{r}}} < \frac{c_\mathrm{r}}{a_\mathrm{r}}, \quad \frac{\mu_{1\mathrm{r}}'}{\mu_{1\mathrm{r}}} < \frac{b_\mathrm{r}}{a_\mathrm{r}}, \quad \frac{\mu_{2\mathrm{r}}'}{\mu_{1\mathrm{r}}} < \frac{d_\mathrm{r}}{a_\mathrm{r}}, \quad \frac{\mu_{3\mathrm{r}}'}{\mu_{1\mathrm{r}}} < \frac{d_\mathrm{r}}{a_\mathrm{r}} \tag{C.86}$$

(In deriving these, we have assumed that $\mu_{1\mathrm{r}} > 0$; but this is a requirement for stability anyway.)

C.4.2 Two non-zero amplitudes

Standing rolls (SRo)

When two amplitudes are non-zero there are three different cases, depending on which of the amplitudes are taken to be non-zero. If two 'opposite' amplitudes, e.g. z_1 and w_1, are non-zero, then we have standing rolls (SRo).

After setting $z_2 = z_3 = w_2 = w_3 = 0$, we obtain:

$$\dot{z}_1 = \mu_1 z_1 + a|z_1|^2 z_1 + b|w_1|^2 z_1 \tag{C.87}$$

$$\dot{w}_1 = \mu_1' w_1 + a|w_1|^2 w_1 + b|z_1|^2 w_1 \tag{C.88}$$

(These are unchanged even when linearized perturbations to z_2, z_3, w_2 and w_3 are allowed.) This corresponds to system 2 (page 217).

The linearized equations for perturbations to the other four variables decouple into two groups:

$$\delta\dot{z}_2 = (\mu_2 + c|z_1|^2 + d|w_1|^2)\delta z_2 + f z_1 w_1 \delta\bar{w}_2 \tag{C.89}$$

$$\delta\dot{w}_2 = (\mu_2' + c|w_1|^2 + d|z_1|^2)\delta w_2 + f z_1 w_1 \delta\bar{z}_2 \tag{C.90}$$

and

$$\delta\dot{z}_3 = (\mu_3 + c|z_1|^2 + d|w_1|^2)\delta z_3 + f z_1 w_1 \delta\bar{w}_3 \tag{C.91}$$

$$\delta\dot{w}_3 = (\mu_3' + c|w_1|^2 + d|z_1|^2)\delta w_3 + f z_1 w_1 \delta\bar{z}_3. \tag{C.92}$$

These correspond to system 3 (page 218). See Table C.1.

Travelling rectangles, type 1 (TRe1)

These have z_1 and z_2 non-zero. As before, the equations decouple into three groups. We may first solve for z_1 and z_2, determining stability with respect to perturbations δz_1 and δz_2; we can then consider perturbations to the other variables separately.

The equations for z_1 and z_2 are

$$\dot{z}_1 = \mu_1 z_1 + a|z_1|^2 z_1 + c|z_2|^2 z_1 \tag{C.93}$$

$$\dot{z}_2 = \mu_2 z_2 + a|z_2|^2 z_2 + c|z_1|^2 z_2 \tag{C.94}$$

Equation system	Parameters	Results
2	$A = \mu_1$, $B = \mu'_1$, $C = a$, $D = b$	The solutions for z_1 (from x) and w_1 (from y), plus two stability eigenvalues
3	$A = \mu_2 + c\lvert z_1\rvert^2 + d\lvert w_1\rvert^2$, $B = f\lvert z_1\rvert\lvert w_1\rvert$, $C = \mu'_2 + c\lvert w_1\rvert^2 + d\lvert z_1\rvert^2$, $\omega = \mathrm{d}/\mathrm{d}t(\arg z_1 + \arg w_1)$	Two stability eigenvalues
3	$A = \mu_3 + c\lvert z_1\rvert^2 + d\lvert w_1\rvert^2$, $B = f\lvert z_1\rvert\lvert w_1\rvert$, $C = \mu'_3 + c\lvert w_1\rvert^2 + d\lvert z_1\rvert^2$, $\omega = \mathrm{d}/\mathrm{d}t(\arg z_1 + \arg w_1)$	Two stability eigenvalues

Table C.1: *Existence and stability calculation for SRo. The first column refers to equation systems from section C.2, which are to be solved given the parameter values in the second column. The third column shows what to do with the results. For example, in this case, system 2 is to be solved to determine whether the solution exists, and if so, the values of x and y (from equations C.13–C.16) will give the appropriate solutions for z_1 and z_2 respectively. Also, two stability eigenvalues may be obtained from system 2, and four from system 3, as indicated; the solution is stable if the real parts of all of these eigenvalues are negative.*

Equation system	Parameters	Results
2	$A=\mu_1$, $B=\mu_2$, $C=a$, $D=c$	The solutions for z_1 (from x) and z_2 (from y), plus two stability eigenvalues
1	$A=\mu_1'+b\|z_1\|^2+d\|z_2\|^2$, $B=f\|z_1\|\|z_2\|$, $C=\mu_2'+b\|z_2\|^2+d\|z_1\|^2$, $\omega=\mathrm{d}/\mathrm{d}t(\arg z_2-\arg z_1)$	Two stability eigenvalues
The other two stability eigenvalues are: $\mu_3+c(\|z_1\|^2+\|z_2\|^2)$ and $\mu_3'+d(\|z_1\|^2+\|z_2\|^2)$.		

Table C.2: *Existence and stability calculation for TRe1.*

This is equivalent to system 2 (page 217).

The perturbations $\delta\dot{z}_3$ and $\delta\dot{w}_3$ evolve according to

$$\delta\dot{z}_3 = [\mu_3+c(|z_1|^2+|z_2|^2)]\delta z_3 \tag{C.95}$$

$$\delta\dot{w}_3 = [\mu_3'+d(|z_1|^2+|z_2|^2)]\delta w_3. \tag{C.96}$$

These can both be solved trivially, providing two more stability eigenvalues.

The final two equations needed are

$$\delta\dot{w}_1 = (\mu_1'+b|z_1|^2+d|z_2|^2)\delta w_1 + f\bar{z}_1 z_2\delta w_2 \tag{C.97}$$

$$\delta\dot{w}_2 = (\mu_2'+b|z_2|^2+d|z_1|^2)\delta w_2 + f z_1\bar{z}_2\delta w_1 \tag{C.98}$$

This corresponds to system 1 (page 217). See Table C.2.

Travelling rectangles, type 2 (TRe2)

These arise when z_1 and w_3 are nonzero. The equations for z_1 and w_3 themselves decouple, and are

$$\dot{z}_1 = \mu_1 z_1 + a|z_1|^2 z_1 + d|w_3|^2 z_1 \tag{C.99}$$

$$\dot{w}_3 = \mu_3' w_3 + a|w_3|^2 w_3 + d|z_1|^2 w_3. \tag{C.100}$$

This corresponds to system 2 (page 217).

The equations for small perturbations δz_2 and δw_2 are

$$\delta\dot{z}_2 = (\mu_2+c|z_1|^2+d|w_3|^2)\delta z_2 \tag{C.101}$$

$$\delta\dot{w}_2 = (\mu_2'+c|w_3|^2+d|z_1|^2)\delta w_2 \tag{C.102}$$

Equation system	Parameters	Results
2	$A = \mu_1$, $B = \mu_3'$, $C = a$, $D = d$	The solutions for z_1 (from x) and w_3 (from y), plus two stability eigenvalues
1	$A = \mu_3 + b\|w_3\|^2 + c\|z_1\|^2$, $B = f\|z_1\|\|w_3\|$, $C = \mu_1' + b\|z_1\|^2 + c\|w_3\|^2$, $\omega = \mathrm{d}/\mathrm{d}t(\arg z_1 - \arg w_3)$	Two stability eigenvalues
The other two stability eigenvalues are: $\mu_2 + c\|z_1\|^2 + d\|w_3\|^2$ and $\mu_2' + c\|w_3\|^2 + d\|z_1\|^2$.		

Table C.3: *Existence and stability calculation for TRe2.*

These can be solved trivially, giving two further stability eigenvalues.

The equations for δz_3 and δw_1 are

$$\delta \dot{z}_3 = (\mu_3 + b|w_3|^2 + c|z_1|^2)\delta z_3 + f z_1 \bar{w}_3 \delta w_1 \tag{C.103}$$

$$\delta \dot{w}_1 = (\mu_1' + b|z_1|^2 + c|w_3|^2)\delta w_1 + f \bar{z}_1 w_3 \delta z_3 \tag{C.104}$$

This corresponds to system 1 (page 217). See Table C.3.

C.4.3 Three non-zero amplitudes

Oscillating triangles (OT)

These are found when z_1–z_3 are nonzero. (Of course, there is a similar solution with w_1–w_3 nonzero, which can be found by exchanging z_j and w_j, and μ_j and μ_j'.)

The equations for z_1–z_3 decouple from the linearized equations for δw_1–δw_3. The former set of equations is as follows:

$$\dot{z}_1 = \mu_1 z_1 + a|z_1|^2 z_1 + c(|z_2|^2 + |z_3|^2) z_1 \tag{C.105}$$

$$\dot{z}_2 = \mu_2 z_2 + a|z_2|^2 z_2 + c(|z_3|^2 + |z_1|^2) z_2 \tag{C.106}$$

$$\dot{z}_3 = \mu_3 z_3 + a|z_3|^2 z_3 + c(|z_1|^2 + |z_2|^2) z_3. \tag{C.107}$$

This corresponds to system 5 (page 218).

Equation system	Parameters	Results
5	$A = \mu_1$, $B = \mu_2$, $C = \mu_3$, $D = a$, $E = c$	The solutions for z_1 (from x), z_2 (from y) and z_3 (from z), plus three stability eigenvalues
4	$A = \mu_1' + b\|z_1\|^2 + d(\|z_2\|^2 + \|z_3\|^2)$, $B = \mu_2' + b\|z_2\|^2 + d(\|z_3\|^2 + \|z_1\|^2)$, $C = \mu_3' + b\|z_3\|^2 + d(\|z_1\|^2 + \|z_2\|^2)$, $\alpha_1 = f\|z_2\|\|z_3\|$, $\alpha_2 = f\|z_1\|\|z_3\|$, $\alpha_3 = f\|z_1\|\|z_2\|$, $\omega_1 = \mathrm{d}/\mathrm{d}t(\arg z_1)$, $\omega_2 = \mathrm{d}/\mathrm{d}t(\arg z_2)$, $\omega_3 = \mathrm{d}/\mathrm{d}t(\arg z_3)$	Three stability eigenvalues

Table C.4: *Existence and stability calculation for OT.*

The equations for the δw's are:

$$\delta\dot{w}_1 = (\mu_1' + b|z_1|^2 + d(|z_2|^2 + |z_3|^2))\delta w_1 + f\bar{z}_1 z_2 \delta w_2 + f\bar{z}_1 z_3 \delta w_3 \quad (C.108)$$

$$\delta\dot{w}_2 = (\mu_2' + b|z_2|^2 + d(|z_3|^2 + |z_1|^2))\delta w_2 + f z_1 \bar{z}_2 \delta w_1 + f\bar{z}_2 z_3 \delta w_3 \quad (C.109)$$

$$\delta\dot{w}_3 = (\mu_3' + b|z_3|^2 + d(|z_1|^2 + |z_2|^2))\delta w_3 + f z_1 \bar{z}_3 \delta w_1 + f z_2 \bar{z}_3 \delta w_2. \quad (C.110)$$

This corresponds to system 4 (page 218). See Table C.4.

New solution branch

As mentioned in Chapter 3, there is an additional solution to these equations (not found by Roberts et al. 1986), which is found by choosing z_1, z_2 and w_3 to be non-zero. The equations for these three quantities are then

$$\dot{z}_1 = \mu_1 z_1 + a|z_1|^2 z_1 + c|z_2|^2 z_1 + d|w_3|^2 z_1 \quad (C.111)$$

$$\dot{z}_2 = \mu_2 z_2 + a|z_2|^2 z_2 + c|z_1|^2 z_2 + d|w_3|^2 z_2 \quad (C.112)$$

$$\dot{w}_3 = \mu_3' w_3 + a|w_3|^2 w_3 + d(|z_1|^2 + |z_2|^2) w_3 \quad (C.113)$$

This corresponds to our system 6 (page 219).

Equation system	Parameters	Results
6	$A = \mu_1$, $B = \mu_2$, $C = \mu_3'$, $D = a$, $E = c$, $F = d$	The solutions for z_1 (from x), z_2 (from y) and w_3 (from z), plus three stability eigenvalues
4	$A = \mu_1' + b\|z_1\|^2 + c\|w_3\|^2 + d\|z_2\|^2$, $B = \mu_2' + b\|z_2\|^2 + c\|w_3\|^2 + d\|z_1\|^2$, $C = \mu_3 + b\|w_3\|^2 + c(\|z_1\|^2 + \|z_2\|^2)$, $\alpha_1 = f\|z_2\|\|w_3\|$, $\alpha_2 = f\|z_1\|\|w_3\|$, $\alpha_3 = f\|z_1\|\|z_2\|$, $\omega_1 = \mathrm{d}/\mathrm{d}t(\arg z_1)$, $\omega_2 = \mathrm{d}/\mathrm{d}t(\arg z_2)$, $\omega_3 = \mathrm{d}/\mathrm{d}t(\arg w_3)$	Three stability eigenvalues

Table C.5: *Existence and stability calculation for the new solution branch (section C.4.3).*

The linearized equations for perturbations to the other three variables are

$$\delta\dot{z}_3 = (\mu_3 + b|w_3|^2 + c(|z_1|^2 + |z_2|^2))\delta z_3 + f z_1 \bar{w}_3 \delta w_1 + f z_2 \bar{w}_3 \delta w_2 \quad \text{(C.114)}$$

$$\delta\dot{w}_1 = (\mu_1' + b|z_1|^2 + c|w_3|^2 + d|z_2|^2)\delta w_1 + f \bar{z}_1 z_2 \delta w_2 + f \bar{z}_1 w_3 \delta z_3 \quad \text{(C.115)}$$

$$\delta\dot{w}_2 = (\mu_2' + b|z_2|^2 + c|w_3|^2 + d|z_1|^2)\delta w_2 + f \bar{z}_2 w_3 \delta z_3 + f \bar{z}_2 z_1 \delta w_1 \quad \text{(C.116)}$$

This corresponds to our system 4 (page 218). See Table C.5.

C.4.4 Four non-zero amplitudes

Only certain combinations are possible here. For example, if $w_2 = w_3 = 0$, but the other four amplitudes are nonzero, then an inspection of the equations reveals that $\dot{w}_2$ and $\dot{w}_3$ would be nonzero in general, so these two amplitudes would immediately become nonzero, which is a contradiction. More generally, we cannot have three z's and one w, or three w's and one z, nonzero; we must have two of each being nonzero.

Without loss of generality, therefore, we can assume that $z_1 z_2 \neq 0$ and $z_3 = 0$. For $\dot{z}_3$ to be zero we would require $w_3 = 0$ (the other possibility is $w_1 = w_2 = 0$, but by assumption we must have four nonzero amplitudes, not three). We conclude that z_1, z_2, w_1 and w_2 are nonzero in this case. (All other cases with four nonzero amplitudes can now be generated by cyclic permutations.)

The equations for the four nonzero amplitudes in this case are

$$\begin{aligned}
\dot{z}_1 &= \mu_1 z_1 + a|z_1|^2 z_1 + b|w_1|^2 z_1 + c|z_2|^2 z_1 + d|w_2|^2 z_1 + f z_2 w_2 \bar{w}_1 \qquad (C.117)\\
\dot{z}_2 &= \mu_2 z_2 + a|z_2|^2 z_2 + b|w_2|^2 z_2 + c|z_1|^2 z_2 + d|w_1|^2 z_2 + f z_1 w_1 \bar{w}_2 \qquad (C.118)\\
\dot{w}_1 &= \mu_1' w_1 + a|w_1|^2 w_1 + b|z_1|^2 w_1 + c|w_2|^2 w_1 + d|z_2|^2 w_1 + f z_2 w_2 \bar{z}_1 \qquad (C.119)\\
\dot{w}_2 &= \mu_2' w_2 + a|w_2|^2 w_2 + b|z_2|^2 w_2 + c|w_1|^2 w_2 + d|z_1|^2 w_2 + f z_1 w_1 \bar{z}_2 \qquad (C.120)
\end{aligned}$$

These can be re-written as amplitude and phase equations:

$$\begin{aligned}
\dot{R}_1 &= (\mu_{1\mathrm{r}} + a_\mathrm{r} R_1^2 + b_\mathrm{r} S_1^2 + c_\mathrm{r} R_2^2 + d_\mathrm{r} S_2^2) R_1 + |f| \cos(\arg f - \psi) R_2 S_2 S_1 \qquad (C.121)\\
\dot{R}_2 &= (\mu_{2\mathrm{r}} + a_\mathrm{r} R_2^2 + b_\mathrm{r} S_2^2 + c_\mathrm{r} R_1^2 + d_\mathrm{r} S_1^2) R_2 + |f| \cos(\arg f + \psi) R_1 S_1 S_2 \qquad (C.122)\\
\dot{S}_1 &= (\mu_{1\mathrm{r}}' + a_\mathrm{r} S_1^2 + b_\mathrm{r} R_1^2 + c_\mathrm{r} S_2^2 + d_\mathrm{r} R_2^2) S_1 + |f| \cos(\arg f - \psi) S_2 R_2 R_1 \qquad (C.123)\\
\dot{S}_2 &= (\mu_{2\mathrm{r}}' + a_\mathrm{r} S_2^2 + b_\mathrm{r} R_2^2 + c_\mathrm{r} S_1^2 + d_\mathrm{r} R_1^2) S_2 + |f| \cos(\arg f + \psi) S_1 R_1 R_2 \qquad (C.124)\\
\dot{\psi} &= \mu_{1\mathrm{i}} - \mu_{2\mathrm{i}} + \mu_{1\mathrm{i}}' - \mu_{2\mathrm{i}}'\\
&\quad + (a_\mathrm{i} + b_\mathrm{i} - c_\mathrm{i} - d_\mathrm{i})(R_1^2 + S_1^2 - R_2^2 - S_2^2)\\
&\quad + |f| \sin(\arg f - \psi) \left(\frac{R_2 S_1 S_2}{R_1} + \frac{R_1 R_2 S_2}{S_1} \right)\\
&\quad - |f| \sin(\arg f + \psi) \left(\frac{R_1 S_1 S_2}{R_2} + \frac{R_1 R_2 S_1}{S_2} \right). \qquad (C.125)
\end{aligned}$$

Here $R_j = |z_j|$ and $S_j = |w_j|$ $(j = 1, 2)$, and $\psi = \arg z_1 - \arg z_2 + \arg w_1 - \arg w_2$. The value of ψ determines which of two possible solution types occurs: $\psi = 0$ corresponds to standing rectangles (SRe), and $\psi = \pi$ corresponds to wavy rolls of the first kind (WR1).

The other two equations needed are

$$\begin{aligned}
\delta\dot{z}_3 &= [\mu_3 + c(|z_1|^2 + |z_2|^2) + d(|w_1|^2 + |w_2|^2)]\delta z_3 + f(z_1 w_1 + z_2 w_2)\delta\bar{w}_3 \qquad (C.126)\\
\delta\dot{w}_3 &= [\mu_3' + c(|w_1|^2 + |w_2|^2) + d(|z_1|^2 + |z_2|^2)]\delta w_3 + f(z_1 w_1 + z_2 w_2)\delta\bar{z}_3 \qquad (C.127)
\end{aligned}$$

These equations can all be placed into AUTO to determine existence/stability for particular cases.

C.4.5 Five non-zero amplitudes

This is impossible. For example, if $z_1 = 0$ but all the other amplitudes are nonzero, it is clear from the equations that $\dot{z}_1$ will be nonzero, which contradicts the assumption that z_1 will be zero for all time.

C.4.6 Six non-zero amplitudes

In this case we have the full complexity of equations (C.72)–(C.77) to deal with. We can break the system down into amplitude and phase equations by writing $z_j = R_j \exp(\mathrm{i}\theta_j)$, $w_j = S_j \exp(\mathrm{i}\phi_j)$. The amplitude equations give the following:

$$\begin{aligned} \dot{R}_1 &= \mu_{1\mathrm{r}} R_1 + (a_\mathrm{r} R_1^2 + b_\mathrm{r} S_1^2 + c_\mathrm{r}(R_2^2 + R_3^2) + d_\mathrm{r}(S_2^2 + S_3^2))R_1 \\ &\quad + |f| \cos(\arg f + \psi_3) R_2 S_2 S_1 + |f| \cos(\arg f - \psi_2) R_3 S_3 S_1 \end{aligned} \tag{C.128}$$

$$\begin{aligned} \dot{S}_1 &= \mu'_{1\mathrm{r}} S_1 + (a_\mathrm{r} S_1^2 + b_\mathrm{r} R_1^2 + c_\mathrm{r}(S_2^2 + S_3^2) + d_\mathrm{r}(R_2^2 + R_3^2))S_1 \\ &\quad + |f| \cos(\arg f + \psi_3) S_2 R_2 R_1 + |f| \cos(\arg f - \psi_2) S_3 R_3 R_1 \end{aligned} \tag{C.129}$$

with the equations for $\dot{R}_2$ and $\dot{R}_3$ being obtained by cyclic permutation. Here $\psi_1 = \arg z_3 + \arg w_3 - \arg z_2 - \arg w_2$; ψ_2 and ψ_3 are defined by cyclic permutations of this. The three ψ_j add up to zero so only two of them need to be kept track of at any one time. The evolution equation for ψ_1 is

$$\begin{aligned} \dot{\psi}_1 &= \mu_{3\mathrm{i}} + \mu'_{3\mathrm{i}} - \mu_{2\mathrm{i}} - \mu'_{2\mathrm{i}} \\ &\quad + a_\mathrm{i}(R_3^2 + S_3^2 - R_2^2 - S_2^2) + b_\mathrm{i}(S_3^2 + R_3^2 - S_2^2 - R_2^2) \\ &\quad + c_\mathrm{i}(R_2^2 + S_2^2 - R_3^2 - S_3^2) + d_\mathrm{i}(S_2^2 + R_2^2 - S_3^2 - R_3^2) \\ &\quad + |f| \left(\frac{R_1 S_1 S_3}{R_3} + \frac{S_1 R_1 R_3}{S_3} \right) \sin(\arg f + \psi_2) + |f| \left(\frac{R_2 S_2 S_3}{R_3} + \frac{S_2 R_2 R_3}{S_3} \right) \sin(\arg f - \psi_1) \\ &\quad - |f| \left(\frac{R_3 S_3 S_2}{R_2} + \frac{S_3 R_3 R_2}{S_2} \right) \sin(\arg f + \psi_1) - |f| \left(\frac{R_1 S_1 S_2}{R_2} + \frac{S_1 R_1 R_2}{S_2} \right) \sin(\arg f - \psi_3) \end{aligned} \tag{C.130}$$

and again, the equations for $\dot{\psi}_2$ and $\dot{\psi}_3$ can be obtained by cyclic permutation. Depending on the values of these phase variables, we can produce either standing hexagons, standing regular triangles, twisted rectangles, or wavy rolls of the second kind.

The equations in this form can be analysed using the program AUTO.